Differential Geometry IV: General relativity

10.1 Let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth solution to the wave equation

$$\Box_\eta \psi = 0 \tag{1}$$

on $(\mathbb{R}^{n+1}, \eta)$. For any $\tau \in \mathbb{R}$, we will denote by $\mathcal{E}[\psi](\tau)$ the total energy of ψ at time $t = \tau$, i.e.

$$\mathcal{E}[\psi](\tau) \doteq \int_{t=\tau} \Big((\partial_t \psi)^2 + |\nabla_x \psi|^2 \Big)\, dx.$$

(a) Using the energy inequality for the domain of a truncated cone that we established in class, show that, if the initial data $(\psi_0, \psi_1) = \big(\psi|_{t=0}, \partial_t \psi_{t=0}\big)$ of ψ at $t = 0$ are compactly supported, then for any $\tau \in \mathbb{R}$ we have

$$\mathcal{E}[\psi](\tau) = \mathcal{E}[\psi](0) = \int_{t=0} \Big(\psi_1^2 + |\nabla_x \psi_0|^2 \Big)\, dx.$$

(b) Show that the same equality is still true if (ψ_0, ψ_1) are not necessarily compactly supported, but satisfy

$$\int_{t=0} \Big(\psi_1^2 + |\nabla_x \psi_0|^2 \Big)\, dx < +\infty.$$

(Hint: Apply the energy identity on a sequence of truncated cones of increasingly large radius R. In order to estimate the corresponding integral on the conical side $\mathcal{S}$ of the domain ans show that it goes to 0 $R \to +\infty$, apply the energy identity on a different but appropriately chosen domain which is contained in $\{\frac{1}{2}R \leqslant |x| \leqslant R\}$.)

10.2 (a) Let $T > 0$ be a positive number. Show that there exists some $C_T > 0$ depending only on T such that the following estimate holds for any solution ψ of (1) on $(\mathbb{R}^{n+1}, \eta)$ with compactly supported initial data at $\{t = 0\}$:

$$\sup_{\tau \in [0,T]} \Big(\mathcal{E}[\psi](\tau) + \int_{t=\tau} \psi^2\, dx \Big) \leqslant C_T \cdot \Big(\mathcal{E}[\psi](0) + \int_{t=0} \psi^2\, dx \Big). \tag{2}$$

Note that the above estimate should be viewed as an energy-type inequality which includes lower order terms.

(Hint: In order to prove the inequality for the second term in the left hand side, start by using the expression $\int_{t=\tau} \psi^2\, dx = \int_0^\tau \int_{t=s} \partial_t(\psi^2)\, dx ds + \int_{t=0} \psi^2\, dx$. Then apply a Cauchy–Schwarz inequality for the spacetime integral and use Gronwal's inequality.)

*(b) For any integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we will denote its Fourier transform by $\hat{f}$, i.e.

$$\hat{f}(\xi) \doteq \int_{\mathbb{R}^n} e^{2\pi i \langle \xi, x \rangle} f(x)\, dx.$$

For a function ψ on $\mathbb{R}^{n+1}$, we will similarly denote by $\hat{\psi}$ its Fourier transform with respect to the *space* variables x. If ψ solves (1), show that $\hat{\psi}(t,\xi)$ solves the following ODE in time:

$$\partial_t^2 \hat{\psi}(t,\xi) + |\xi|^2 \hat{\psi}(t,\xi) = 0.$$

Deduce that, in terms of the initial data (ψ_0, ψ_1) at $t = 0$, ψ can be expressed via the relation

$$\hat{\psi}(t,\xi) = \cos(|\xi|t)\hat{\psi}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{\psi}_1(\xi).$$

Using the above expression, show that, in the case when the space dimension satisfies $n \geqslant 2$, for any solution ψ arising from smooth and compactly supported initial data which satisfy $\psi_0 = 0$ and $\hat{\psi}_1(0) \neq 0$, we have

$$\limsup_{\tau \to \pm\infty} \int_{t=\tau} |\psi|^2 \, dx = +\infty.$$

In particular, the estimate (2) cannot hold with a constant C_T which is uniformly bounded in $T > 0$ (unlike the simplest energy identity).

10.3 Consider the wave-type equation

$$\Box_\eta \phi + A^\mu \partial_\mu \phi + B\phi = 0$$

on $\mathbb{R}^{n+1}$, with A a smooth vector field on $\mathbb{R}^{n+1}$ and $B \in C^\infty(\mathbb{R}^{n+1})$. For any $0 < T \leqslant R$, consider the truncated spacetime cone

$$\Omega_{T,R} = \bigcup_{\tau \in [0,T]} \{\tau\} \times B_{R-\tau}.$$

Show that the modified energy

$$\widetilde{\mathcal{E}}[\phi](\tau) = \int_{\{t=\tau\}\times B_{R-\tau}} \Big((\partial_t \phi)^2 + |\nabla_x \phi|^2 + |\phi|^2 \Big) \, dx$$

satisfies the analogue of (2), i.e.

$$\widetilde{\mathcal{E}}[\phi](\tau) \leqslant C_{T,R} \widetilde{\mathcal{E}}[\phi](0).$$

for some $C_{T,R} \geqslant 0$ depending only on T, R and the precise form of A, B. Deduce that $\phi = 0$ on the whole of $K_{T,R}$ if $(\phi, \partial_t \phi)|_{\{0\}\times B_R} = 0$.
(*Hint: Starting from the energy identity as we did in class for* $\Box_\eta \phi = 0$*, show that* $\widetilde{\mathcal{E}}[\phi](\tau)$ *satisfies a Gronwall-type inequality; see also Ex. 10.2.*)

10.4 Let us consider the Schwarzschild exterior spacetime $(\mathcal{M}, g_M)$ (i.e. region I of the maximal extension) in the (t^*, r, θ, ϕ) coordinate system of Exercise 8.2 (with $r \in (2M, +\infty)$). Recall, that these coordinates are smooth across the future event horizon and the Schwarzschild metric takes the form

$$g_M = -\Big(1 - \frac{2M}{r}\Big)(dt^*)^2 + \frac{4M}{r}dt^*dr + \Big(1 + \frac{2M}{r}\Big)dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big).$$

(a) Show that the hypersurfaces $\{t^* = \text{const}\}$ are spacelike and compute their future directed unit normal vector field $\hat{n}$. Setting

$$u = e^{2M\log(r-2M)-t^*},$$

show also that the hypersurfaces $\{u = \epsilon\}$, for $\epsilon > 0$ sufficiently small, are spacelike. What is the hypersurface $\{u = 0\}$? For $T, \epsilon > 0$, sketch the domain $\{u \geqslant \epsilon\} \cap \{0 \leqslant t^* \leqslant T\}$ on the Penrose diagram.

(b) Let ψ be a smooth solution of the wave equation $\Box_{g_M}\psi = 0$. Compute the coordinate expression of the energy flux $\mathcal{E}[\psi](\tau) = \int_{\{t^*=\tau\}} J^{(\partial_{t^*})}[\psi]_\mu \hat{n}^\mu$.

(c) Show that $\mathcal{E}[\psi](\tau)$ is non-increasing in τ. *(Hint: Apply the divergence identity for div$J^{(\partial_{t^*})}$ in regions of the form $\{\tau_1 \leqslant t^* \leqslant \tau_2\} \cap \{u > \epsilon\}$ as $\epsilon \to 0$.)*